Irrationality Without Number Theory

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1. Introduction

Let k and t be positive integers. It is well known that $k^{1/t}$ is either a positive integer or an irrational number. This can be proved easily from the unique factorization theorem.

An interesting question is how much number theory is necessary in order to prove this theorem. Maier and Niven [2] and Floyd [1] have presented proofs of this theorem that use no facts about prime numbers. The former use the division algorithm and induction to simplify and generalize Steinhaus's proof [5, pages 38-39] of the case $k = t = 2$; the latter uses the Euclidean algorithm to simplify and generalize Sagher's proof [4] of the case $t = 2$. When $t = 2$, Maier and Niven's technique is especially interesting because it does not explicitly use any number theory, only very basic inequalities.

In this paper, we also prove the theorem for all k and t. Although our proof is more complicated than Floyd's, it has the advantage of not explicitly using any number theory, so it can be presented to a very general audience. Our proof is simpler than Maier and Niven's. To accustom the reader to the technique involved, we begin by presenting a very simple proof for the case $t = 2$. The idea is due to a proof presented by Niven [3] for the case $k = t = 2$.

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Proposition 1. *$k^{1/2}$ is either a positive integer or an irrational number.*

Proof: Let $x = k^{1/2}$ and assume that x is rational. Then choose the smallest positive integer n such that nx is an integer. Let

$$n' = n(x - \lfloor x \rfloor).$$

where $\lfloor x \rfloor$ denotes the integer part (floor) of x. Since $0 \leq x - \lfloor x \rfloor < 1$, it follows that $0 \leq n' < n$. Note that $n' = nx - n\lfloor x \rfloor$, which is an integer. What's more,

$$n'x = nx^2 - (nx)\lfloor x \rfloor,$$

which is an integer. Because n is the smallest positive integer with that property, n' must be 0, so $x - \lfloor x \rfloor = 0$. In other words x is an integer. ∎

2. Main Results

Theorem 2. *$k^{1/t}$ is either an integer or an irrational number.*

Proof: Let $x = k^{1/t}$, and assume that x is rational. Then choose the smallest positive integer n such that nx is an integer. For $0 \leq i \leq t-1$, we make the following claim, which we prove by induction on i:

Inductive Hypothesis: $n^{t-i-1}x^{t-i}$ is an integer.

The base case ($i = 0$) is obvious. The theorem follows from the final case ($i = t-1$). It remains to establish the inductive step. Assume that the inductive hypothesis is true for $i-1$ so that

$$n^{t-i}x^{t-i+1} \text{ is an integer.}$$

Let

$$z = n^{t-i-1}x^{t-i}.$$

We complete the induction by showing that z is an integer. Let

$$n' = n(z - \lfloor z \rfloor).$$

Then $0 \leq n' < n$. We note that n' is the difference of two integers because $nz = (nx)^{t-i}$, which is an integer by the initial assumption, and $n\lfloor z \rfloor$ is the product of two integers. Therefore n' is a nonnegative integer. What's more,

$$\begin{aligned} n'x &= nzx - n\lfloor z \rfloor x \\ &= n^{t-i}x^{t-i+1} - (nx)\lfloor z \rfloor, \end{aligned}$$

which is the difference of two integers (by the inductive hypothesis and the initial assumption), hence an integer. Since n is the smallest positive integer with that property, n' must be 0. Therefore $z - \lfloor z \rfloor$ is 0, so z is an integer, completing the induction. ∎

Using a trick from [2], we can extend this proof directly to apply to all algebraic integers.

Corollary 3. *If x is an algebraic integer then x is either a rational integer or an irrational number.*

Proof: Recall that an algebraic integer is by definition a zero of a monic polynomial with integer coefficients, *i.e.*,

$$x^t + \sum_{0 \leq i \leq t-1} c_i x^i = 0.$$

Assume that x is rational, and choose the smallest positive integer n such that nx is an integer. Then, as observed by Maier and Niven

$$\begin{aligned} n^{t-1}x^t &= -\sum_{0 \leq i \leq t-1} c_i n^{t-1} x^i \\ &= -\sum_{0 \leq i \leq t-1} c_i n^{t-1-i} (nx)^i, \end{aligned}$$

which is an integer because nx is an integer. This establishes the base case for the induction in the preceding theorem. The remainder of the proof is exactly as before, so we conclude that x is an integer. ∎

3. Generalizations

Floyd also shows that $k^{1/3}$ cannot be a root of a quadratic equation with rational coefficients unless $k^{1/3}$ is an integer. Using a different technique (but no number theory) we extend this to $k^{1/t}$ for all odd t. (The result is false for even t because $(2^{t/2})^{1/t}$ is a root of a quadratic equation with rational coefficients.)

Corollary 4. *Let t be odd. If $k^{1/t}$ is a root of a quadratic equation with rational coefficients then $k^{1/t}$ is an integer.*

Proof: Let $x = k^{1/t}$ and assume that x is a root of a quadratic equation with rational coefficients. By the quadratic formula, there exists a rational number p and a nonnegative rational number q such that $x = p \pm q^{1/2}$. If $q = 0$, then x is rational, and hence an integer by Theorem 2, so we need only consider the case $q > 0$.

$$\begin{aligned} k &= (p \pm q^{1/2})^t \\ &= \sum_i \binom{t}{i} p^{t-i} q^{i/2} (\pm 1)^i \qquad \text{by the binomial theorem} \\ &= \pm \sum_{i \text{ odd}} \binom{t}{i} p^{t-i} q^{\lfloor i/2 \rfloor} q^{1/2} + \sum_{i \text{ even}} \binom{t}{i} p^{t-i} q^{i/2} \\ &= \pm s q^{1/2} + r, \end{aligned}$$

where

$$s = \sum_{i \text{ odd}} \binom{t}{i} p^{t-i} q^{\lfloor i/2 \rfloor},$$

and

$$r = \sum_{i \text{ even}} \binom{t}{i} p^{t-i} q^{i/2}.$$

Therefore $q^{1/2}$ is equal to the rational number $\pm(k - r)/s$, unless $s = 0$. Each term of s is nonnegative because p is raised to an even power and q is positive. Furthermore, the last term ($i = t$) of s is equal to $q^{\lfloor t/2 \rfloor}$, which is strictly positive. Therefore s is positive, so $q^{1/2}$ is rational, so x is rational. By Theorem 2, x is an integer. ∎

It is natural to ask whether $k^{1/t}$ can be a non-integral zero of an mth degree polynomial with rational coefficients. (If t has a divisor d such that $2 \le d \le m$, then the answer is yes, because $\left(2^{t/d}\right)^{1/t}$ is a zero of $x^{m-d}(x^d-2)$.) The preceding corollary resolves the special case $m=2$. It is unclear whether the proof technique works when $m=3$ or $m=4$; however it is clear that the proof technique cannot be applied when $m \ge 5$, because there is no closed-form solution of the quintic equation. Because the statement of the general case requires some number theory, it seems likely that a resolution of the general case will require some number theory. We present an approach that works when $t=5$ and $m=3$.

Corollary 5. *If $k^{1/5}$ is a root of a cubic equation with rational coefficients then $k^{1/5}$ is an integer.*

Proof: Let $x = k^{1/5}$ and assume that x is a root of a cubic equation with rational coefficients. Then there exist rational numbers p, q, and r such that $x^3+px^2+qx+r=0$. Identically,

$$x^5-k = (x^3+px^2+qx+r)(x^2-px+(p^2-q))+(2pq-p^3-r)x^2+(pr+q^2-p^2q)x+qr-p^2r-k.$$

Since $x^5-k=0$ and $x^3+px^2+qx+r=0$,

$$(2pq-p^3-r)x^2+(pr+q^2-p^2q)x+qr-p^2r-k=0.$$

Thus x satisfies a quadratic equation with rational coefficients, and we are done by the preceding corollary, unless each coefficient of the quadratic equation is zero. From the first coefficient, we find $r=2pq-p^3$. Substituting into the second coefficient, we find $q^2+p^2q-p^4=0$. By the quadratic formula, $q=\frac{1}{2}(-1\pm 5^{1/2})p^2$. Therefore, $5^{1/2} = \pm(2q/p^2+1)$, which is rational (and hence an integer by the preceding proposition), unless $p=0$. But if $p=0$, then $q=0$ and $r=0$, so $x^3=0$. Thus x is an integer. ∎

4. Problems

Prove, using a minimum amount of number theory:

- If t is not divisible by 3 and $k^{1/t}$ is a zero of an irreducible cubic polynomial with integer coefficients, then $k^{1/t}$ must be an integer.

- If m and t are relatively prime and $k^{1/t}$ is a zero of an irreducible mth degree polynomial with integer coefficients, then $k^{1/t}$ must be an integer.

Serge Lang posed a problem similar to the latter on his Galois theory final at Yale in 1989. In a hint, he suggests that one look at the constant term in the mth degree polynomial, expressed as a product of tth roots of k, and deduce that k must be a tth power. We leave the details as a pleasant exercise, and we encourage the reader to look for a solution to either problem using less mathematics.

5. Discussion

In our proofs, we used induction and the floor operation $\lfloor x \rfloor$. Since x is assumed to be rational, that operation implicitly uses the division algorithm. Thus, although the presentation of our proofs is simple, the depth of mathematics implicitly used is the same as in [2].

In Maier and Niven's proofs, they used induction, the division algorithm, and the floor operation. Although they appear not to use any number theory in the case $t = 2$, they use the floor operation in that case. In fact, it would be very surprising if any proof of irrationality really used no number theory.

In Floyd's proofs, he used the Euclidean algorithm, which can be derived by using induction and the division algorithm. Thus, Floyd's proofs also use the same depth of mathematics as in our proofs and in [2].

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